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RESEARCH MEMORANDUM

THE DECOMPOSITION ALGORITHM FOR LINEAR PROGRAMMING

Notes on Linear Programming and Extensions — Part 57

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SUMMARY

A procedure is presented for the efficient computational solution of linear programs having a certain structural property characteristic of a large class of problems of practical interest. This property makes possible the decomposition of the problem into a sequence of small linear programs whose iterated solutions solve the given problem through a generalization of the simplex method for linear programming.

CONTENTS

THE DECOMPOSITION ALGORITHM FOR LINEAR PROGRAMMING[1]

1. THE DECOMPOSED LINEAR PROGRAM

Many linear programming problems of practical interest have the property of being composed of separate linear programming problems, which are tied together by a number of constraints that is considerably less than the total number imposed on the problem. When the suitably-ordered matrix of coefficients of such a problem is displayed in the usual way, a pattern emerges like that shown in Fig. 1. In this figure the constraint matrix has

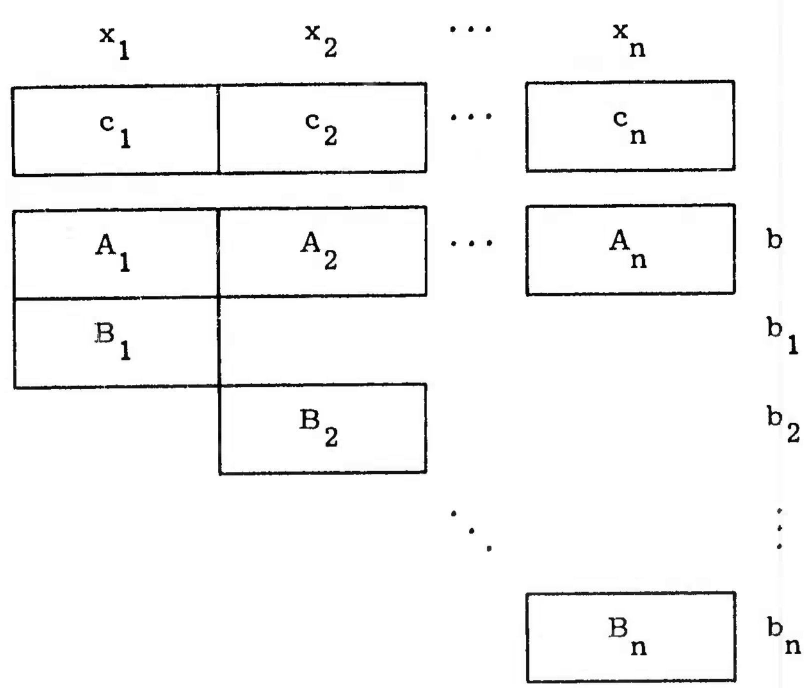

Fig. 1 — The original problem

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[1] A preliminary version of this research memorandum appeared originally under the title A Decomposition Principle for Linear Programs, The RAND Corporation, Paper P-1544, November 10, 1958, and was presented at The RAND Symposium on Mathematical Programming in March, 1959.

been partitioned into non-zero blocks A_j and B_j; the "right-hand side" column of constants, $b, b_1, \ldots, b_n$, is arrayed correspondingly; and the "costs" — the coefficients of the objective form — are partitioned into the blocks $c_1, c_2, \ldots, c_n$.

More particularly, suppose that we have the following:

an m-vector b,

$$\left.\begin{array}{l} A_j, \text{ an } m \text{ by } n_j \text{ matrix,} \\ B_j, \text{ an } m_j \text{ by } n_j \text{ matrix,} \\ c_j, \text{ an } n_j\text{-vector, and} \\ x_j, \text{ a variable } n_j\text{-vector,} \end{array}\right\} \text{for each } j = 1, \ldots, n.$$

Then the problem illustrated in Fig. 1 is posed as a linear programming problem in $\Sigma_j n_j$ variables, subject to $m + \Sigma_j m_j$ constraints. It can be summarized as follows:

<u>The Decomposed Program.</u> Find the vectors x_j $(j = 1, \ldots, n)$ satisfying the constraints

(1) $$\Sigma_j A_j x_j = b \text{ and } x_j \geq 0 \qquad \text{(for all } j\text{)},$$

(2) $$B_j x_j = b_j \qquad \text{(for all } j\text{)},$$

which minimize the linear form

(3) $$\Sigma_j c_j x_j .$$

It would seem that each of the n sets of constraints of (2) constitutes a subproblem of secondary importance to the whole program, and that these sets should be studied mainly through the restrictions they impose on the activities of the joint constraints (1). Pursuing this point of view leads us from the decomposed program to the equivalent extremal problem of the next section. For the sake of simplicity in formulating the extremal problem, it is assumed that

$$(4) \qquad S_j = \{x_j \mid x_j \geq 0,\ B_j x_j = b_j\}$$

is bounded for each j. Only minor changes, which will be discussed in Sec. 4, are required in order to remove this restriction in the treatment which follows.

The results of this paper arise from the statement of the decomposed program as a "generalized linear programming problem" in n variables, in which the m—element column of coefficients associated with each variable is drawn freely from the convex set S_j instead of being fixed as in the ordinary linear programming problem. This generalized problem is developed more fully elsewhere [5].

2. THE EXTREMAL PROBLEM

In this section we formulate a type of linear programming problem in which those features of the decomposed program (1—3) that pertain to its joint constraints are brought to the fore. In the new problem, the extreme

points of the sets defined by the relations (2) will yield the data for the problem.

For $j = 1, \ldots, n$, let $W_j = \{x_{j1}, \ldots, x_{jK_j}\}$ be the set of all extreme points of the convex polyhedron S_j defined by the conditions $x_j \geq 0$, $B_j x_j = b_j$; also define

$$\text{(5)} \qquad \left.\begin{aligned} P_{jk} &= A_j x_{jk} \\ c_{jk} &= c_j x_{jk} \end{aligned}\right\} \text{ for } k = 1, \ldots, K_j.$$

2.1. The Extremal Program

Find numbers s_{jk} $(j = 1, \ldots, n;\ k = 1, \ldots, K_j)$, satisfying

$$\text{(6)} \qquad \Sigma_{j,k} P_{jk} s_{jk} = b, \qquad s_{jk} \geq 0 \qquad \text{(for all } j,\ k\text{)},$$

and

$$\text{(7)} \qquad \Sigma_k s_{jk} = 1 \qquad \text{(for all } j\text{)},$$

which minimize the linear form

$$\text{(8)} \qquad \Sigma_{j,k} c_{jk} s_{jk}.$$

The relation of the extremal problem to the original problem lies in the fact that, since S_j is a bounded, convex polyhedral set, any point x_j of S_j may be written as a convex combination of its extreme points, that is, as $\Sigma_k x_{jk} s_{jk}$, where $\{s_{j1}, \ldots, s_{jK_j}\}$ satisfy (7); and the expressions (6) and (8) are just the expressions (1) and (3) of the decomposed problem

rewritten in terms of the s_{jk}. This relation is stated in the following lemma, which requires no proof:

Lemma. If the numbers $\{s_{jk}\}$ solve the extremal program (6–8), then the vectors

(9) $$s_j = \Sigma_k x_{jk} s_{jk}, \qquad j = 1, \ldots, n,$$

solve the problem (1–3).

Note that, by (5) and (9),

$$\Sigma_j A_j x_j = \Sigma_j \Sigma_k A_j x_{jk} s_{jk} = b \quad \text{and} \quad \Sigma_j c_j x_j = \Sigma_{j,k} c_{jk} s_{jk}.$$

The matrix of coefficients for the extremal problem is displayed in Fig. 2. Its constraint equations are m + n in number; the m joint

$$
\begin{array}{ll|ccc|ccc|c|ccc|c}
 & & s_{11} & \cdots & s_{1K_1}, & s_{21} & \cdots & s_{2K_2} & ,\ldots, & s_{n1} & \cdots & s_{nK_n} & \\
 & & c_{11} & \cdots & c_{1K_1} & c_{21} & \cdots & c_{2K_2} & ,\ldots, & c_{n1} & \cdots & c_{nK_n} & \\
 & & & \text{columns} & & & \text{columns} & & & & \text{columns} & & \\
 & \pi & P_{11} & \cdots & P_{1K_1} & P_{21} & \cdots & P_{2K_2} & ,\ldots, & P_{n1} & \cdots & P_{nK_n} & b \\
\text{Prices} & \bar{\pi} & 1 & \cdots & 1 & & & & & & & & 1 \\
 & & & & & 1 & \cdots & 1 & & & & & 1 \\
 & & & & & & & & \ddots & & & & \vdots \\
 & & & & & & & & & 1 & \cdots & 1 & 1
\end{array}
$$

Fig. 2 — The extremal problem

constraints of the original problem have gone over into the m constraints (6) constituting the upper block in Fig. 2, and the m_j constraints of the j^{th} subproblem have gone over into single constraints of the form (7). The reduction in the total number of constraints is sizable in case the m_j are large, and it is on this fact that the computational efficiency of the decomposition algorithm relies. The reduction appears to have been accomplished, however, by greatly enlarging the number of variables in the problem from the original $\Sigma_j n_j$ to the number $\Sigma_j K_j$; the proposed method would be of little interest or value if it were not possible effectively to reduce this number.

3. THE DECOMPOSITION ALGORITHM

The central idea of the decomposition algorithm is that the extremal linear programming problem of (6—8) and Fig. 2 can be solved by the simplex method for linear programming without prior calculation of all the data given in the statement of the problem. Typically, only a small number of the $\Sigma_j K_j$ variables present will ever play an active role in the course of solving the problem, and as will be seen, the coefficients needed in handling just these variables may be generated at the time they are to come under consideration. The principle of coefficient generation was first employed in the study of multi-commodity network flows [3].

Let us repeat here the essential features of the simplex method that will be used. The phenomenon of degeneracy plays the same role in the

extremal problem that it does in any linear programming problem, and standard methods [1] can be invoked to handle it when necessary, so that it need not be discussed here. The problem discussed in the next section — that of finding a first feasible solution to the extremal problem — is similar to the same problem for a general linear program. It will be assumed that the usual data are at hand for performing one typical step of the simplex method on the extremal problem.

Since the extremal problem has $m + n$ equation constraints, there will be at hand $m + n$ columns ($(m + n)$–component vectors corresponding to extreme points) that constitute a "feasible basis." These columns are linearly independent, and the unique solution of the constraint equations (6, 7), which is obtained by setting to zero those variables associated with all other columns, is nonnegative. If the revised simplex method is used in performing the calculations, there will also be at hand the $(m + n)$–vector of "prices" $(\pi; \bar{\pi})$ — the m–vector π being associated with the first m constraints and the n–vector $\bar{\pi}$ with the remaining n, as in Fig. 2. In the general linear program, the inner product of the price vector with any column of the basis must equal the cost associated with that column. In the case of the extremal problem, if we let $\bar{\pi}_j$ be the j^{th} component of $\bar{\pi}$, this relationship can be written as

$$(10) \qquad \pi P_{jk} + \bar{\pi}_j = c_{jk}$$

for basic columns drawn from the j^{th} partition, $j = 1, \ldots, n$.

One step of the simplex method iteration for solving the extremal problem would be performed as follows: Find a column of the constraint matrix whose "reduced cost" is negative, that is, for which

$$c_{jk} - \pi P_{jk} - \bar{\pi}_j < 0 . \tag{11}$$

(Commonly, the column which minimizes the reduced cost is chosen.) Add this column to the current basis, and delete one column from the basis in such a way that the new basis is still feasible. If no column satisfying (11) can be found, then the current solution $\{s_{jk}\}$ solves the extremal problem. Otherwise, the simplex method gives the appropriate rules for the removal of a column from the basis, and for the calculation of the new prices $(\pi; \bar{\pi})$ associated with the new basis with which the next iterative step can begin.

A procedure for applying the simplex algorithm to the extremal problem without having all the data of that problem at hand can now be stated. It is supposed that only the $m + n$ columns of a feasible basis for the problem are given.

3.1. The Decomposition Algorithm: Iterative Step

Given $m + n$ columns of the form $(P_{jk}; 0, \ldots, 1, \ldots, 0)$ that constitute a feasible basis for the extremal problem, their associated costs c_{jk}, and prices $(\pi; \bar{\pi})$ satisfying (10); for each $j = 1, \ldots, n$, let $\bar{x}_j$ be an extreme point of S_j minimizing the linear form

(12) $(c_j - \pi A_j)x_j$, under the constraints $x_j \geq 0$, $B_j x_j = b_j$;

and let $\bar{x}_{j_0}$ be such that

(13) $$\delta = (c_{j_0} - \pi A_{j_0})\bar{x}_{j_0} - \bar{\pi}_{j_0} = \mathrm{Min}_j \left[(c_j - \pi A_j)\bar{x}_j - \bar{\pi}_j\right].$$

If $\delta < 0$, form the new column and its associated cost for the extremal problem as

(14) $(A_{j_0}\bar{x}_{j_0}; 0, \ldots, 1, \ldots, 0)$ and $c_{j_0}\bar{x}_{j_0}$;

add this column to the basis, and form a new basis and new prices using the rules of the simplex method. If $\delta \geq 0$, terminate the algorithm; $\{s_{jk}\}$ solves the extremal problem, and the relations (9) give the solution of the original problem.

Theorem. The decomposition algorithm terminates in a finite number of iterations, yielding a solution of the extremal problem.

Proof. Since the termination of the simplex method as applied to the general linear programming problem in a finite number of iterations is known [1], it is sufficient to show that the rules of the iterative step of the decomposition algorithm yield a column satisfying the criterion (11), if this is possible. For the column defined by (14), the left-hand side of (11) is simply $c_{j_0}\bar{x}_{j_0} - \pi A_{j_0} - \pi_{j_0} = \delta$, which is as small as possible. (The lemma of Sec. 2 shows that a solution of the original problem is

obtained when a solution of the extremal problem has been found by this algorithm.)

In summary, a cycle of the decomposition algorithm can be stated this way:

Given $m + n$ columns constituting a feasible basis for the extremal problem, use the prices associated with that basis to form the modified costs, $c_j - \pi A_j$, for each of the subproblems.

Minimize $(c_j - \pi A_j)x_j$ for $x_j \geq 0$, $B_j x_j = b_j$. If $\delta < 0$, so that the original problem is not solved, then from an appropriate subproblem solution, construct a new column for the extremal problem and form a new feasible basis incorporating that column, deleting another column from the basis in accordance with the rules of the simplex method.

4. DETAILS ON USE OF THE ALGORITHM

This section considers some details regarding use of the decomposition algorithm: how to get the algorithm started; how to deal with unbounded solutions of the subproblems, which may occur when the boundedness restriction is removed; some variations of the selection technique (12) and of methods for decomposing a problem.

4.1. Initiating the Algorithm

The decomposition algorithm can be started with precisely the same device, called Phase One, that is used for the general linear programming problem. This device consists in augmenting the problem with $m + n$

"artificial" variables in terms of which an initial basic feasible solution, and the prices associated with the corresponding initial feasible basis, are readily found. The decomposition algorithm can then be applied to the problem of removing the artificial variables. After this has been done, the required initial conditions for the ordinary application of the algorithm are automatically met.

For $i = 1, \ldots, m + n$, let y_i be a nonnegative variable; let I_i be the i^{th} column of the $(m + n)$–order identity matrix; and let $c_i = 1$ be the cost associated with the variable y_i. For Phase One, replace all the costs $\{c_j\}$ of the original problem with zero vectors. It can be assumed without loss of generality that the right–hand side vectors b, $\{b_j\}$ are all non–negative.

Designating $\{I_i\}$ as the initial feasible basis, employ the decomposition algorithm in the minimization of the linear form $\Sigma_i y_i$. [Note that the initial feasible solution and the initial prices are given by $(y_1, \ldots, y_{m+n}) = (b; b_1, \ldots, b_n)$ and $(\pi; \bar{\pi}) = (1, \ldots, 1)$.]

If the form $\Sigma_i y_i$ cannot be reduced to zero, then the extremal problem has no feasible solution; if the form can be reduced to zero, then a feasible solution is a fortiori at hand. Typically, this Phase One will have ended with none of the artificial columns left in the basis. (It occasionally happens that an artificial column remains in the basis although its variable is, of course, zero. Handling this case requires the use of an additional constraint, described elsewhere [1], in the problem.) The cost vectors of the original

problem are then restored, and Phase Two, the application of the algorithm to the proper extremal problem, can proceed as in Sec. 3.

4.2. Extension to the Unbounded Case

It was assumed in (4) that the constraint set S_j of each subproblem was bounded. If this is not the case, then it may happen in applying the decomposition algorithm that an unbounded solution is obtained for the subproblem:

(15) Minimize $(c_j - \pi A_j)x_j$ under the constraints $x_j \geq 0$, $B_j x_j = b_j$.

The only change incurred in the algorithm by this is that the rule for forming a new column for the extremal problem must be extended. An unbounded solution of the problem (15) will cause the simplex method to yield a vector y_j such that

(16) $$y_j \geq 0, \quad B_j y_j = 0, \quad \text{and} \quad (c_j - \pi A_j)y_j < 0 .$$

The relationships (16) give the direction of an infinite ray of feasible solutions along which the cost form of (15) proceeds to $-\infty$.

In this case, the column and cost to be added to the extremal problem have the form

(17) $$(A_j y_j;\ 0,\ \ldots,\ 0) \quad \text{and} \quad c_j y_j ,$$

instead of the form (14). The "1" has been omitted from the column so that the variable s_{jk} associated with this column will not be constrained

by the relation $\Sigma_k s_{jk} = 1$, imposed upon the columns created from bounded solutions. Any nonnegative multiple of this column is admissible, corresponding to the fact that any nonnegative multiple of the vector y_j is admissible in the representation of the constraint set of (15) as the sum of a bounded polyhedron and a polyhedral convex cone. It is easy to check that with this extra freedom the decomposition algorithm still consists only of applying the simplex method to the extremal problem thus extended, and hence the algorithm has the same termination properties as in the bounded case, since the number of possible rays generated by the simplex solutions of the subproblems is finite.

4.3. Variations of the Decomposition Process

It is possible to pursue the decomposition algorithm in different ways for the sake of computational efficiency. One suggestion is easily made: retain as many of the solutions $\bar{x}_j$ to the subproblems (12) as would provide new basic columns for the extremal problem; that is, for each j such that

$$(c_j - \pi A_j)\bar{x}_j < \bar{\pi}_j ,$$

form the column $(A_j\bar{x}_j; 0, \ldots, 1, \ldots, 0)$ and its cost $c_j\bar{x}_j$, and adjoin all these to the given basic columns, employing the simplex method to find a minimum among this extended set of columns. This procedure, while requiring perhaps more simplex-method iterations, should require fewer subprogram solutions to be found.

It should be noted that finding the n subproblem solutions of (12) at each iteration of this procedure may not require the complete solution of n linear programs. If the prices in the extremal problem do not change much from one iteration to the next, then the new subproblem solutions may not differ much, if at all, from their former ones; in that case, the new solutions can be found with little labor. Also, in many practical cases, the subproblems themselves have such structure that their solutions are readily found by means of special devices — for example, when the subproblems are transportation problems.

Another variation in the use of decomposition depends on the fact that a given linear programming problem may be decomposed to various degrees. For example, in the original problem (1–3) pictured in Fig. 1, the first two partitions might have been considered collectively as one, embracing the single vector $(x_1; x_2)$. Little would be changed in the description of the decomposition algorithm, except that the first subproblem of the family (12) would read as follows:

(18) Minimize $(c_1 - \pi A_1)x_1 + (c_2 - \pi A_2)x_2$ under the restrictions $(x_1, x_2) \geq 0,\ B_1x_1 = b_1,\ B_2x_2 = b_2$;

the column constructed therefrom would be

(19) $(A_1\bar{x}_1 + A_2\bar{x}_2; 0, \ldots, 1, \ldots, 0)$, where $(\bar{x}_1, \bar{x}_2)$ solves (18).

The two variables of the problem (18), however, are completely independent,

so that its solution $(\bar{x}_1; \bar{x}_2)$ is just composed of the solutions $\bar{x}_1$, $\bar{x}_2$, of the two separate problems of the ordinary form for $j = 1, 2$. The result, then, of "aggregating" the first two partitions has been to aggregate their resultant columns according to the formula (19).

If this aggregation is carried as far as possible, all the n subproblems may be taken together as one. On account of the independence of the n parts of this one subproblem, finding the subproblem solution involves the same work as before. An advantage, however, lies in the fact that the extremal problem now has only $m + 1$ constraints, instead of $m + n$, and thus is easier to handle. The decomposition of a linear programming problem into a problem having only one partition, when applied to the classical transportation problem, yields an algorithm which promises considerable efficiency in the case of transportation from a small number of origins to a large number of destinations [4].

Finally, it should be mentioned that the linearity of the cost functions $c_j x_j$ for the partitions of the problem is not essential for the application of the decomposition algorithm. Suppose instead that in each partition this function were replaced by the function $f_j(x_j)$, which need only be convex for success of the method. The decomposition algorithm would be unchanged, except that the subproblems (12) would assume the form

(20) Minimize $f_j(x_j) - \pi A_j x_j$ under the constraints $x_j \geq 0$, $B_j x_j = b_j$.

Solutions $\bar{x}_j$ of these problems would then be used to construct new extremal

columns by formula (14) with associated costs $f_j(\bar{x}_j)$. In the nonlinear case, termination of the decomposition algorithm is not assured, but convergence to a solution of the problem is. The arguments used differ from those above; this extension of the decomposition algorithm is given elsewhere [5].

5. A NUMERICAL EXAMPLE

The small problem we shall solve by means of the decomposition algorithm is somewhat artificial, but it will suffice to show the utility of the method even in the case of a single partition.

A homogeneous commodity is to be transported from each of two sources to each of four destinations. The costs of transporting one unit of the commodity from a given source to a given destination appear as entries in the matrix of Fig. 3. Beside the matrix and below it are written,

source \ destination	1	2	3	4	supply
1	3	6	6	5	9
2	8	1	3	6	8
demand	2	7	3	5	

Fig. 3 — Data for the example

respectively, the amounts of the commodity available from each source and demanded at each destination. One requirement is added which prevents this from being an ordinary transportation problem: it is required that the sum of three times the quantity shipped from source 1 to destination 3 and twice the quantity shipped from source 2 to destination 2 be 10.

In order to formulate this as a linear programming problem, denote by t_{ij} $(i = 1, 2;\ j = 1, \ldots, 4)$ the quantity to be shipped from source i to destination j. The coefficients of all the constraints are shown in Fig. 4.

x	t_{11}	t_{12}	t_{13}	t_{14}	t_{21}	t_{22}	t_{23}	t_{24}		
c	3	6	6	5	8	1	3	6		
A			3			2			10	b
	1	1	1	1					9	
					1	1	1	1	8	
	1				1				2	
B		1				1			7	b_1
			1				1		3	
				1				1	5	

Fig. 4 — Decomposition of the example

Since there is only one partition, subscripts are not needed to describe the parts of the matrix. The row constituting the matrix A represents the joint

constraint on t_{13} and t_{22}. The block B consists of six equations asserting that the net flow from each source must be just the amount available, and that the net flow to each destination must be what is demanded.

A has only one row, and the problem has only one partition, so that $m = 1$ and $n = 1$. The extremal problem is thus a two-equation problem, and the price vector $(\pi; \bar{\pi})$ has two components. Of course, $\{t_{ij}\} = x$. The single subproblem of (12), the problem of minimizing $(c - \pi A)x$ under the constraints $Bx = b$, becomes here just that of solving the transportation problem shown in Fig. 5, since the subproblem constraints are precisely those of the transportation problem of Fig. 3. Rapid special methods for

3	6	$6 - 3\pi$	5	9
8	$1 - 2\pi$	3	6	8
2	7	3	5	

Fig. 5 — The subproblem as a transportation problem

solving transportation problems are well known [2]; we need not go into them here.

Beginning with a basic feasible solution for the extremal problem constructed from the artificial columns I_1, I_2 (see Sec. 4), the major data generated in the decomposition algorithm solution of this problem are given in the table below. We have omitted only the inverse of each feasible basis, which is generally carried along in simplex method calculations.

DECOMPOSITION ALGORITHM ITERATIONS FOR THE EXAMPLE

Iteration number, k	Basic solution	π	$\overline{\pi}$	Subproblem solution, x	Extremal cost	Extremal column, P_k	δ
1	$10\,I_1 + I_2$	1	1	$\boxed{\begin{matrix} 1 & & 3 & 5 \\ 1 & 7 & & \end{matrix}}$	0	$\begin{bmatrix} 23 \\ 1 \end{bmatrix}$	-24
2	$\frac{10}{23} P_1 + \frac{13}{23} I_2$	$-\frac{1}{23}$	1	$\boxed{\begin{matrix} 2 & 7 & & \\ & & 3 & 5 \end{matrix}}$	0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	-1
3	$\frac{10}{23} P_1 + \frac{13}{23} P_2$	0	0	End of Phase One			
	[Phase Two begins with costs for P_1, P_2: 61, 87.]						
3 Cont.		$-\frac{26}{23}$	87	$\boxed{\begin{matrix} 2 & & 2 & 5 \\ & 7 & 1 & \end{matrix}}$	53	$\begin{bmatrix} 20 \\ 1 \end{bmatrix}$	$-11\frac{9}{23}$
4	$\frac{1}{2} P_3 + \frac{1}{2} P_2$	$-\frac{17}{10}$	87	$\boxed{\begin{matrix} 2 & 2 & & 5 \\ & 5 & 3 & \end{matrix}}$	57	$\begin{bmatrix} 10 \\ 1 \end{bmatrix}$	-13
5	$0\,P_3 + 1\,P_4$	$-\frac{2}{5}$	61	$\boxed{\begin{matrix} 2 & 2 & & 5 \\ & 5 & 3 & \end{matrix}}$			0

NOTE: Since $\delta \geq 0$, the basic solution is optimal. The objective function has the values $75\frac{16}{23}$, 70, 57 in the three stages of Phase Two.

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